Exercise 27

Solve the differential equation using the method of variation of parameters.

$$y'' - 2y' + y = \frac{e^x}{1+x^2}$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2e^{rx}$$

Plug these formulas into equation (1).

$$r^2e^{rx} - 2(re^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 1 = 0$$

Solve for r.

$$(r-1)^2 = 0$$

$$r = \{1\}$$

Two solutions to the ODE are e^x and xe^x. By the principle of superposition, then,

$$y_c(x) = C_1e^x + C_2xe^x.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 2y_p' + y_p = \frac{e^x}{1+x^2} \tag{2}$$

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_1(x)e^x + C_2(x)xe^x$$

Differentiate it with respect to x.

$$y_p' = C_1'(x)e^x + C_2'(x)xe^x + C_1(x)e^x + C_2(x)(x+1)e^x$$

If we set

$$C_1'(x)e^x + C_2'(x)xe^x = 0, \tag{3}$$

then

$$y_p' = C_1(x)e^x + C_2(x)(x+1)e^x.$$

Differentiate it with respect to x once more.

$$y_p'' = C_1'(x)e^x + C_2'(x)(x+1)e^x + C_1(x)e^x + C_2(x)(x+2)e^x$$

Substitute these formulas into equation (2).

$$[C_1'(x)e^x + C_2'(x)(x+1)e^x + \cancel{C_1(x)e^x} + \cancel{C_2(x)(x+2)e^x}] - 2[\cancel{C_1(x)e^x} + \cancel{C_2(x)(x+1)e^x}] + [\cancel{C_1(x)e^x} + \cancel{C_2(x)xe^x}] = \frac{e^x}{1+x^2}$$

Simplify the result.

$$C_1'(x)e^x + C_2'(x)(x+1)e^x = \frac{e^x}{1+x^2} \tag{4}$$

Subtract the respective sides of equations (3) and (4) to eliminate $C_1'(x)$.

$$C_2'(x)e^x = \frac{e^x}{1+x^2}$$

Solve for $C_2'(x)$.

$$C_2'(x) = \frac{1}{1+x^2}$$

Integrate this result to get $C_2(x)$, setting the integration constant to zero.

$$C_2(x) = \tan^{-1} x$$

Solve equation (3) for $C_1'(x)$.

$$\begin{aligned} C_1'(x) &= -C_2'(x)x \\ &= -\left(\frac{1}{1+x^2}\right)x \\ &= -\frac{x}{1+x^2} \end{aligned}$$

Integrate this result to get $C_1(x)$, setting the integration constant to zero.

$$\begin{aligned} C_1(x) &= \int^x C_1'(w)\,dw \\ &= -\int^x \frac{w}{1+w^2}\,dw \\ &= -\int^{1+x^2} \frac{1}{u}\left(\frac{du}{2}\right) \\ &= -\frac{1}{2}\int^{1+x^2} \frac{du}{u} \\ &= -\frac{1}{2}\ln|u|\Big|^{1+x^2} \\ &= -\frac{1}{2}\ln(1+x^2) \end{aligned}$$

Therefore,

$$\begin{aligned} y_p &= C_1(x)e^x + C_2(x)xe^x \\ &= \left[-\frac{1}{2}\ln(1+x^2)\right]e^x + (\tan^{-1}x)xe^x \\ &= e^x\left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right] \\ &= e^x\left(x\tan^{-1}x - \ln\sqrt{1+x^2}\right), \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1e^x + C_2xe^x + e^x\left(x\tan^{-1}x - \ln\sqrt{1+x^2}\right), \end{aligned}$$

where C_1 and C_2 are arbitrary constants.